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Shift Automorphisms of the Reduced Free Products of Infinitely Many Algebras

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§0. Introduction.

In the class of C^* -algebras constructed by the reduced free product, the class of the reduced free products of infinitely many copies indexed by the integers \mathbb{Z} is the smallest in the sense of Theorem 1 below.

For such the reduced free product C^* -algebras, the shift automorphism coming from the shift on integers are investigated.

§1. Reduced free products.

The reduced free product construction for C^* -algebras was introduced independently by Avitzour [A] and Voiculescu [V].

By a C^* -probability space (A, ϕ) , we mean that A is a unital C^* -algebra (operator norm closed $*$ -algebra of bounded operators on a Hilbert space) and ϕ is a state of A .

Given a family of C^* -probability space $(A_i, \phi_i)_{i \in I}$, where the GNS representation by each ϕ_i is assumed to be faithful, the reduced free product (A, ϕ) of $(A_i, \phi_i)_{i \in I}$ is denoted by

$$(A, \phi) = \bigstar_{i \in I} (A_i, \phi_i).$$

Let $H_i = L^2(A_i, \phi_i)$, and ξ_i is the image vector of the unit of A_i in H_i . Let $H_i^\circ = H_i \ominus \mathbb{C}\xi_i$. The free product Hilbert space H of $(H_i, \xi_i)_{i \in I}$ is defined for the distinguished vector ξ (which is called the vacuum vector) by the form :

$$H = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq \dots \neq i_n} (H_{i_1}^\circ \otimes \dots \otimes H_{i_n}^\circ), \quad i_j \in I$$

and it is denoted by $(H, \xi) = \bigstar_{i \in I} (H_i, \xi_i)$.

Each A_i acts faithfully on H_i as the left multiplication operators. Let $\overset{\circ}{A}_i = \{x \in A_i; \phi_i(x) = 0\}$, and let

$$\text{red}(A) = \{x_1 x_2 \dots x_n; x_j \in \overset{\circ}{A}_{i_j}, (i_1 \neq i_2 \neq \dots \neq i_n), \}.$$

The A in the reduced free product $(A, \phi) = (A_i, \phi_i)$ is the C^* -algebra on the reduced free product Hilbert space H , and A is generated by the identity operator

on H and $\text{red}(A)$. The ϕ is the state of A , which is called the vacuum state and defined by

$$\phi(a) = \langle a\xi, \xi \rangle.$$

How to operate an $x \in \text{red}(A)$ on H ?

Assume that $x \in \text{red}(A)$ has a form that $x = x_1 x_2 \cdots x_n$, for $x_j \in \overset{\circ}{A}_{i_j}$, ($i_1 \neq i_2 \neq \cdots \neq i_n$). Then

$$x\xi = x_1 \xi_{i_1} \otimes x_2 \xi_{i_2} \otimes \cdots \otimes x_n \xi_{i_n},$$

and for the vector

$$\zeta = \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_m, \quad \zeta_i \in \overset{\circ}{H}_{j_i}, \quad (i = 1, \dots, m),$$

$$x\zeta = x\xi \otimes \zeta, \quad \text{for } i_n \neq j_1,$$

$$x\zeta = (x_1 \zeta_1 - \langle x_1 \zeta_1, \xi_{i_1} \rangle \xi_{i_1}) + \langle x_1 \zeta_1, \xi_{i_1} \rangle \xi, \quad \text{for } i_n = j_1, n = m = 1$$

$$x\zeta = (x_1 \zeta_1 - \langle x_1 \zeta_1, \xi_{i_1} \rangle \xi_{i_1}) \otimes \zeta_2 \otimes \cdots \otimes \zeta_m \\ + \langle x_1 \zeta_1, \xi_{i_1} \rangle \zeta_2 \otimes \cdots \otimes \zeta_m, \quad \text{for } i_n = j_1, n = 1, m \geq 2$$

$$x\zeta = x_1 \xi_{i_1} \otimes \cdots \otimes x_{n-1} \xi_{i_{n-1}} \otimes (x_n \zeta_1 - \langle x_n \zeta_1, \xi_{i_n} \rangle \xi_{i_n}) \\ + \langle x_n \zeta_1, \xi_{i_n} \rangle x_1 \xi_{i_1} \otimes \cdots \otimes x_{n-1} \xi_{i_{n-1}}, \quad \text{for } i_n = j_1, n \geq 2, m = 1$$

and

$$x\zeta = x_1 \xi_{i_1} \otimes \cdots \otimes x_{i_{n-1}} \xi_{i_{n-1}} \otimes (x_n \zeta_1 - \langle x_n \zeta_1, \xi_{i_n} \rangle \xi_{i_n}) \otimes \zeta_2 \otimes \cdots \otimes \zeta_m \\ + \langle x_n \zeta_1, \xi_{i_n} \rangle x_1 \cdots x_{n-1} (\zeta_{j_2} \otimes \cdots \otimes \zeta_{j_m}), \quad \text{for } i_n = j_1, n, m \geq 2.$$

The characterization for the reduced free product of C^* -algebras is given in [V1] as follows:

A given (A, ϕ) is isomorphic to $\ast_{i \in I} (A_i, \phi_i)$ if and only if there exists an isomorphism $\sigma_i : A_i \rightarrow A$ such that

(i) A is generated by $\{\sigma_i(A_i)\}_{i \in I}$,

(ii) $\phi \cdot \sigma_i = \phi_i$,

(iii) $\phi(\sigma_{i_1}(a_{i_1}) \cdots \sigma_{i_n}(a_{i_n})) = 0$ for $a_{i_1} \in \overset{\circ}{A}_{i_1}^\circ$ and $i_1 \neq i_2 \neq \cdots \neq i_n$,

and the GNS construction applied to (A, ϕ) yields a faithful representation of A .

§2. Free products of infinitely many C^* -algebras.

The most typical example of reduced free product C^* -algebra appears as the C^* -algebra generated by the left regular representation for the free product of some groups.

Remember that the free product $\mathbb{Z}_2 \ast \mathbb{Z}_2$ of the group \mathbb{Z}_2 is amenable and not a free group. To get a C^* -algebra belonging to a "new class" from the reduced free product construction $(A_1, \phi_1) \ast (A_2, \phi_2)$, it is often used so called *Arvitzour's unitary condition*:

$$\exists \text{ unitaries } a, b, c \text{ with } a \in \overset{\circ}{A}_1, \quad b, c, b^*c \in \overset{\circ}{A}_2.$$

In the class of C^* -algebras constructed by the reduced free product, the class of the reduced free products of infinitely many copies is the smallest in the following sense:

Theorem 1. Assume that the pair $\{(A_1, \phi_1), (A_2, \phi_2)\}$ satisfies Arvitzour's unitary condition. Then

$$(A_1, \phi_1) * (A_2, \phi_2) \supset \bigstar_{i \in \mathbb{Z}} (B_i, \psi_i),$$

where

$$(B_i, \psi_i) = (A_1, \phi_1) * (A_2, \phi_2), \quad \text{for all } i \in \mathbb{Z}.$$

Let (A_i, ϕ_i) be the copy of a C^* -dynamical system (A_0, ϕ_0) for all $i \in \mathbb{Z}$. The reduced free product $(A, \phi) = \bigstar_{i \in \mathbb{Z}} (A_i, \psi_i)$ has the automorphism α induced by the shift $i \rightarrow i+1$ on the integers \mathbb{Z} . We call such the automorphism α on A *free shift*.

As an example of a more precise inclusion of Theorem 1, we have the following :

Example (cf. [CD]). Let (A_1, ϕ_1) be a non-trivial C^* -dynamical system, that is, $A_1 \neq \mathbb{C}$. Then

$$(A_1, \phi_1) * (C(\mathbb{T}), \int \cdot dt) = \bigstar_{i \in \mathbb{Z}} (B_i, \psi_i) \rtimes_{\alpha} \mathbb{Z} \text{ the crossed product,}$$

for the copy (B_i, ψ_i) of $(A_1, \phi_1) * (C(\mathbb{T}), \int \cdot dt)$, ($i \in \mathbb{Z}$). Here $C(\mathbb{T})$ is the continuous functions on the Torus \mathbb{T} , $\int \cdot dt$ is the state given by the integral, and α is the free shift.

Proposition 2. Let $(A, \phi) = \bigstar_{i \in \mathbb{Z}} (A_i, \psi_i)$, where $(A_i, \phi_i) = (A_0, \phi_0)$ for all $i \in \mathbb{Z}$.

Then the free shift α on A gives an algebraic K -system in the sense of Narnhofer-Thirring [NT], that is, there exists a C^* -subalgebra $B \subset A$ with $B \subset \alpha(B)$, A is generated by the family $\{\alpha^n(B)\}_{n \in \mathbb{Z}}$, and $\bigcap_n \alpha^n(B) = \mathbb{C}1$.

§3. Entropies for the free shift.

By a C^* -dynamical system (A, α, ϕ) , we mean that A is a unital C^* -algebra, α is an automorphism, and ϕ is an α -invariant state of A . For a given C^* -dynamical system (A, α, ϕ) , we denote by $h_{\phi}(\alpha)$ the Connes-Narnhofer-Thirring entropy ([CNT]), by $H_{\phi}(\alpha)$ the Sauvageot-Thouvenot entropy ([ST]), and by $H_{(\phi, \Phi, A_0)}$ Alicki-Fannes entropy ([Af]) which is defined with respect to a globally α -invariant $*$ -subalgebra A_0 of A .

Theorem 3. Let $(A, \phi) = \bigstar_{i \in \mathbb{Z}} (A_i, \psi_i)$, where $(A_i, \phi_i) = (A_0, \phi_0)$ for all $i \in \mathbb{Z}$, and let α be the free shift on A .

(i) ([C1]) Sauvageot-Thouvenot entropy has the relation

$$H_{\phi * \psi}(\alpha * \beta) = H_{\psi}(\beta) = H_{\phi \otimes \psi}(\alpha \otimes \beta),$$

for any C^* -dynamical system (B, ψ, β) . In special, $H_{\phi}(\alpha) = 0$.

(ii) ([CT]) Let $A_0 \subset A$ be the natural α -invariant $*$ -subalgebra generated by the identity and $\text{red}(A)$. Then Alicki-Fannes entropy has

$$H_{(\phi, \alpha, A_0)} = +\infty.$$

In the case where the C^* -algebra is nuclear, Sauvageot-Thouvenot entropy coincides with Connes-Narnhofer-Thirring entropy ([ST]).

The notion of "nuclear" for C^* -algebras corresponds to that of "amenable" for groups. In general, the free product of groups is not amenable, and the reduced free product C^* -algebra is not nuclear.

However there exists a nuclear C^* -algebra which is given as the reduced free product of \mathbb{Z} copies of an algebra. As an example, we have the Cuntz algebra \mathcal{O}_∞ .

Remark. Stormer ([S]) showed that $h_\phi(\alpha) = 0$ for any free shift α .

As another type of a C^* -dynamical entropy, we have a slight modification of Voiculescu's topological entropy $ht(\cdot)$ ([V2]). The definition is as follows :

Definition ([C2]). For a nuclear C^* -algebra A with unity, let $CPA(A)$ be the set of triple (φ, η, C) such that C is a finite dimensional C^* -algebra, and $\varphi : A \rightarrow C$ and $\eta : C \rightarrow A$ are unital completely positive maps.

Let Ω be the set of finite subsets of A and let ϕ be a state of A . For an $\omega \in \Omega$, put

$$scp_\phi(\omega; \delta) = \inf \{ S(\phi \cdot \eta) : (\varphi, \eta, C) \in CPA(A), \|\eta \cdot \varphi(a) - a\| < \delta, a \in \omega \}.$$

Here $S(\phi \cdot \eta)$ means the entropy of the state $\phi \cdot \eta$ of C . For a unital endomorphism ρ of A with $\phi \cdot \rho = \phi$, put

$$ht_\phi(\rho, \omega; \delta) = \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} scp_\phi \left(\bigcup_{i=0}^{N-1} \rho^i(\omega); \delta \right)$$

$$ht_\phi(\rho, \omega) = \sup_{\delta > 0} ht_\phi(\rho, \omega; \delta).$$

Then the entropy $ht_\phi(\rho)$ of ρ is defined by

$$ht_\phi(\rho) = \sup_{\omega \in \Omega} ht_\phi(\rho, \omega).$$

Proposition 4. ([C2]) For a ϕ -preserving automorphism α of A , Connes - Narnhofer - Thirring entropy $h_\phi(\alpha)$, the entropy $ht_\phi(\alpha)$ and Voiculescu's topological entropy $ht(\alpha)$ have in general the following relation :

$$h_\phi(\alpha) \leq ht_\phi(\alpha) \leq ht(\alpha).$$

Remark. ([C2]) We have examples that $h_\phi(\alpha) \neq ht_\phi(\alpha)$ or $ht_\phi(\alpha) \neq ht(\alpha)$.

Since \mathcal{O}_∞ is nuclear, we can apply the entropy $ht_\phi(\alpha)$ for the free shift on \mathcal{O}_∞ .

Proposition 5. Let α be the free shift of \mathcal{O}_∞ . Then

$$ht_\phi(\alpha) = 0,$$

for the vacuum state ϕ which is the unique α -invariant state of \mathcal{O}_∞ .

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